# SQUARING THE PLANE USING ARITHMETIC SEQUENCE GRIDS

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## 1. Introduction

Trying to extend some of the Ponting square packings [1, 3] into a plane tiling, the ones I tried 'almost' work, but don't fit together along a single radial seam. I found related grids that do tile the plane - the first being the 'double rainbow' tiling (Figure 1) - and recently have been studying this type of tiling, made from grids filled with arithmetic sequences of integers. (For instructions on constructing grids, see [2].)

Does every grid tile the plane? What determines whether they do or not? Are there an infinite number of different tilings? Why do the 0-sized squares on the grid always appear in a straight line in the tiling? Which, if any, of the Ponting packings, when extended, tile the plane?

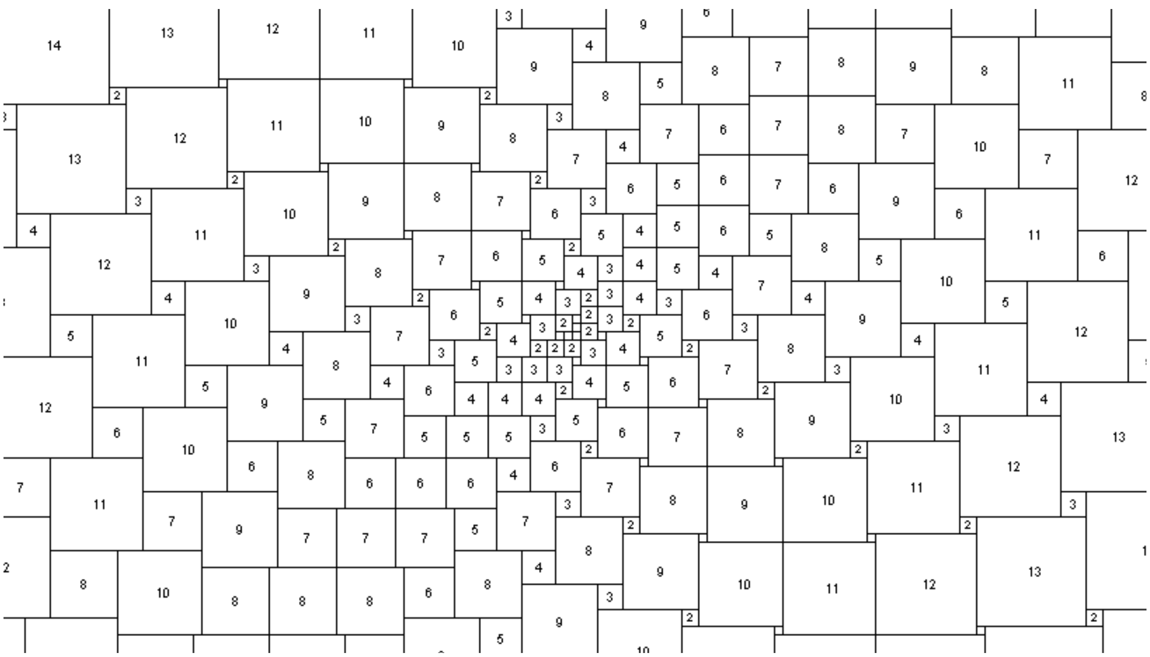


Figure 1. The centre of the 'double rainbow' tiling $\left.\frac{1\quad -1}{\phantom{1\quad -}0}\right| -1$

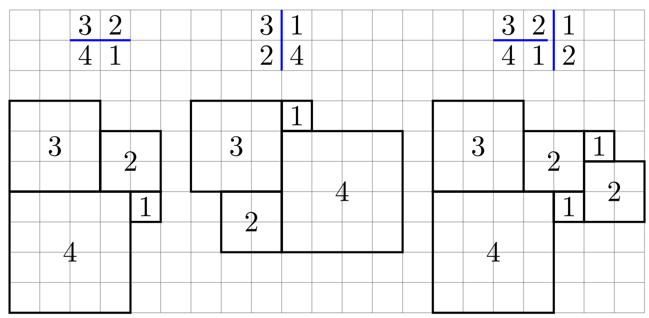
*Date*: June 2017.

## 2. Grids

2.1. **Terms.** An *h-line*, where $a+b=c+d$ : $\dfrac{a\ \ b}{c\ \ d}$

Mostly I'll work with an *hv-line*, a combined h-line/v-line : $\dfrac{a\ \ b}{c\ \ d}\Big|\begin{matrix} e \\ f \end{matrix}$ (Figure 2)
where $a+b=c+d$ and $b+d=e+f$. The $c$ and $f$ will usually be left off, as the four numbers $a$, $b$, $d$ and $e$ are enough to completely specify a grid/tiling. That reduced hv-line can also be used as the name of the grid/tiling.

Figure 2.



An h-line A v-line An hv-line

*A and E.* $A=a-d$ and $E=e-d$, the common differences of the arithmetic sequences, have the same value in every hv-line in a grid.

*Groups.* The grid cells fall into 2 groups – those that can be $a$, $d$ and $e$ in hv-lines, which I'll call *group 1*, and those that are $b$, $c$ and $f$, which I'll call *group 2*.

2.2. **Moving around the grid.** Starting with any hv-line $\dfrac{a\ \ b}{c\ \ d}\Big|\begin{matrix} e \\ f \end{matrix}$, the next hv-line to the north-west will be $\dfrac{a{+}A\ \ b{+}E}{c{+}E\ \ d{+}A}\Big|\begin{matrix} e{+}A \\ f{+}E \end{matrix}$, which I'll also write as $\left[\begin{smallmatrix} +A & +E & +A \\ +E & +A & +E \end{smallmatrix}\right]$. (See Figure 3)

Moving south-east, the changes are equal and opposite: $\left[\begin{smallmatrix} -A & -E & -A \\ -E & -A & -E \end{smallmatrix}\right]$

Similarly, moving north-east: $\left[\begin{smallmatrix} +E & -A & +E \\ -A & +E & -A \end{smallmatrix}\right]$ and south-west: $\left[\begin{smallmatrix} -E & +A & -E \\ +A & -E & +A \end{smallmatrix}\right]$.

The effect of these four motions can be summarized with the notation:

$$\dfrac{\overset{\overset{+A}{\nwarrow}}{a}\quad \overset{\overset{+E\ \ -A}{\nwarrow\ \nearrow}}{b}}{c\qquad d}\ \Bigg|\ \begin{matrix} \overset{\overset{+E}{\nearrow}}{e} \\ f \end{matrix}$$

Movement horizontally or vertically, or to any other hv-line, can be produced by a combination of these elementary operations, e.g.

Move horizontally 1 hv-line to the right = move SE 1 step + move NE 1 step

| | | | | | |
|---|---|---|---|---|---|
| b+2E | e+2A | | a+2E | b-2A | e+2E |
| a+A | b+E | e+A | b-A | e+E | |
| | a | b | e | | |
| a-E | b+A | d | b-E | e-A | |
| b+2A | d-E | | d-A | b-2E | e-2A |

FIGURE 3. Moving around the grid.

*h/v-line sums.* If $T$ is the total sum $a+b+c+d$ of an h-line, the h-line to its right has sum $T-2A$; the h-line below it has sum $T-2E$. The same is true of v-lines.

2.3. **Effect of** $\gcd(A,E)$**.** If $G=\gcd(A,E)>1$, then all values in group 1 are congruent mod $G$, and the values in group 2 are either all congruent to them mod $G$, or none are. E.g.

$$\begin{array}{cc|c} 2 & 1 & 4 \\ \hline 3 & 0 & -3 \end{array} \begin{cases} A=2 \\ E=4 \end{cases} \quad \gcd(A,E)=2 \begin{cases} \text{group 1 even} \\ \text{group 2 odd} \end{cases}$$

$$\begin{array}{cc|c} 5 & 1 & 8 \\ \hline 4 & 2 & -5 \end{array} \begin{cases} A=3 \\ E=6 \end{cases} \quad \gcd(A,E)=3 \begin{cases} \text{group } 1 \equiv 2 \pmod 3 \\ \text{group } 2 \equiv 1 \pmod 3 \end{cases}$$

2.4. **Role of** $A^2+E^2$**.** Adding $\pm(A^2+E^2)$ to one group any number of times moves to another hv-line on the same grid.

Moving north-east $A$ times arrives at the h-line $\begin{bmatrix} a+AE & b-A^2 \\ c-A^2 & d+AE \end{bmatrix}$

Moving south-east $E$ times gets to $\begin{bmatrix} a-AE & b-E^2 \\ c-E^2 & d-AE \end{bmatrix}$

And doing these 2 steps consecutively gets to $\begin{bmatrix} a & b-(A^2+E^2) \\ c-(A^2+E^2) & d \end{bmatrix}$

Similarly, moving north-east $E$ times gets to $\begin{bmatrix} +E^2 & -AE & +E^2 \\ -AE & +E^2 & -AE \end{bmatrix}$; then moving north-west $A$ times gets to $\begin{bmatrix} +E^2+A^2 & = & +E^2+A^2 \\ = & +E^2+A^2 & = \end{bmatrix}$.

2.5. **Finding other grid cell values.** Starting from any group 1 cell ($a,d,e$ in any hv-line) $s$, move right $x$ grid cells and up $y$ grid cells to another group 1 cell ($x+y$ must be even.) This corresponds to moving NE $\frac{x+y}{2}$ steps and SE $\frac{x-y}{2}$ steps. So the new cell value is :

$$s_{x,y} = s + \frac{E(x+y)-A(x-y)}{2} = s + \frac{x(E-A)+y(E+A)}{2}$$

For group 2 cells $(b, c, f)$ :

$$s_{x,y} = s - \frac{A(x+y) + E(x-y)}{2} = s - \frac{x(A+E) + y(A-E)}{2}$$

## 3. Tilings

3.1. **Proper grids and tilings.** Drawing a tiling from a grid, it wraps around the centre twice, so that a half-grid fills 360°, and a whole grid wraps around to 720°, doubly tiling the plane. Most grids tile the plane with two-non-matching layers. Occasionally the two layers match, and a *proper* plane tiling results. (Figure 4)

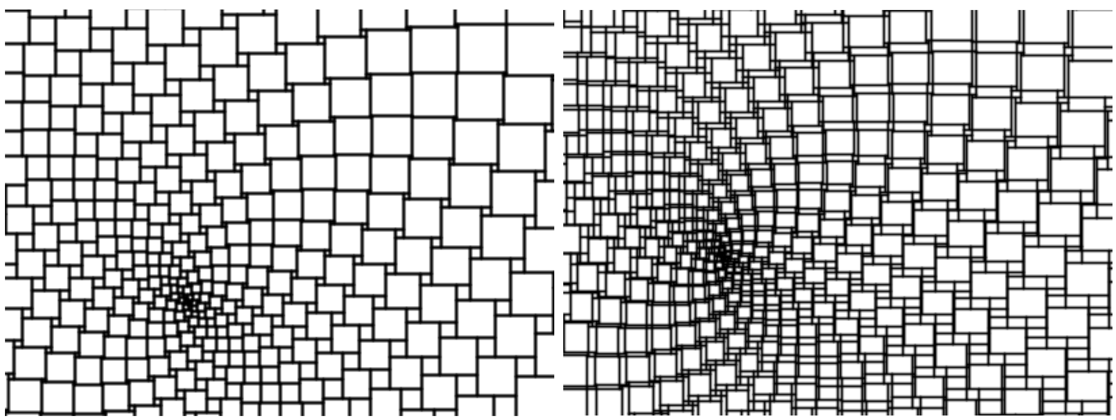


Figure 4. A proper tiling $\frac{8\ \ 73}{0}\Big|^{9}$ and an improper tiling $\frac{8\ \ 63}{0}\Big|^{9}$.

The two layers of a tiling are superimposed on each other perfectly when there is a centre of 180° (C2) rotational *antisymmetry* in the grid, i.e. each square rotates onto another of equal size and opposite sign. Then each half of the grid expands to fill the plane precisely, fitting perfectly along the seam. I'll call these *proper grids.*

3.2. **Centres of rotation and mirrors.** The centre of rotation in a proper grid may be in the centre of a grid square or at a grid vertex; rotation around the midpoint of a side isn't possible. Rotational antisymmetry around these two locations is produced around the grid centre by the two kinds of *mirror*: *0-mirrors* (antisymmetry around a grid 0-square centre) and *quad-mirrors* (antisymmetry around a grid vertex).

3.3. **0-mirrors.** These have at their centre a hv-line of the form $\left.\frac{A\ \ \frac{E-A}{2}}{0}\right|\ E$.

$E - A$ must be even so that $\frac{E-A}{2}$ is an integer. Then $c$ and $f$ are equal and opposite, e.g. in $\frac{3\ \ 1}{4\ \ 0}\Big|\,\frac{5}{-4}$, the 4 and $-4$ squares are not only the same size, but appear in exactly the same place in the tiling. (Figure 5)

If $A = \pm E$ then 0-mirror tilings have D1 symmetry. If $A = -E$, as in the original 'double rainbow' $\frac{1\ \ -1}{0}\Big|^{-1}$, there's mirror symmetry across $y = -x$. If $A = E$, the axis of symmetry is $y = x$. Otherwise, the 0-mirror tilings have no exact symmetry at all, although from a distance they look perfectly symmetrical.

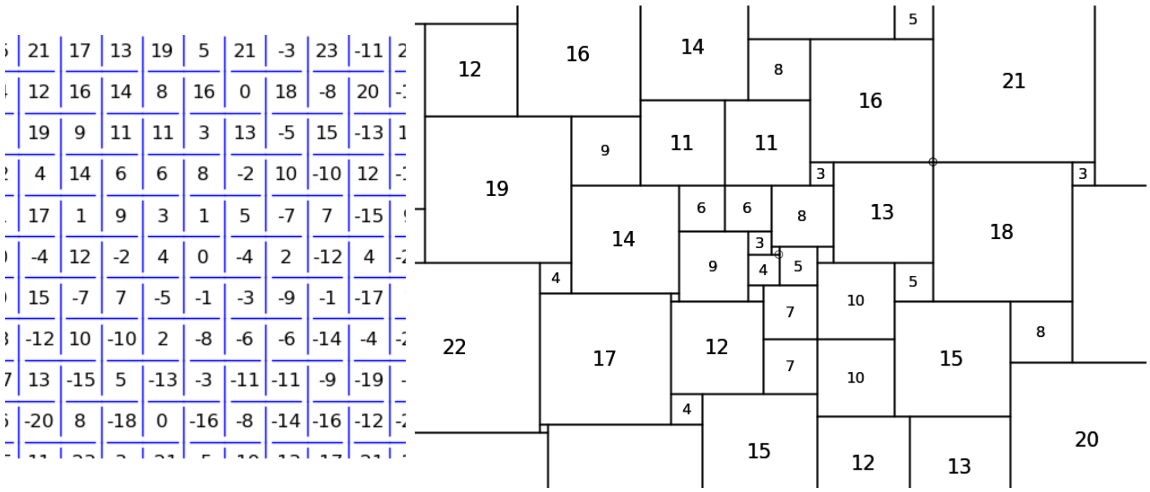



FIGURE 5. A 0-mirror: the centre of the grid for $\left.\frac{3\ \ 1}{0}\right|^{5}$, and its tiling.

3.4. **Quad-mirrors.** A grid with a quad-mirror has a $2 \times 2$ block of $\pm\frac{E}{2}$ cells at its centre. The grid has 90° rotational (C4) antisymmetry. The tiling has 180° rotational (C2) antisymmetry, and a central hv-line with the form :

$$\left.\frac{H{+}A \quad H{-}A}{H}\right|\, 3H$$

where $H = \frac{E}{2}$. (So $E$ must be even.)

E.g. At the centre of the grid for $\left.\frac{5\ \ -1}{2}\right|^{6}$ $(A = 3, E = 4)$ is the block $\begin{smallmatrix} 2 & 2 \\ -2 & -2 \end{smallmatrix}$, which corresponds to the two adjacent 2-squares in the centre of the tiling. (Figure 6)

3.5. **If $E - A$ is odd.** Then evidently $\left.\frac{A \quad \frac{E-A}{2}}{0}\right|\, E$ can't have integer values and a 0-mirror isn't possible. To make a proper tiling from that 'faulty 0-mirror', either:

*a.* Multiply all numbers by 2, e.g. multiply $\left.\frac{4 \quad \frac{1}{2}}{0}\right|\, 5$ by 2 to get the 0-mirror $\left.\frac{8 \quad 1}{9 \quad 0}\right|\, \begin{matrix} 10 \\ -9 \end{matrix}$. (Figure 7), or

*b.* Add $\pm\dfrac{A^2 + E^2}{2}$ to group 2 ($b$ will be $\dfrac{E - A \pm (A^2 + E^2)}{2}$) to get all integers and a proper tiling[1], because a quad-mirror is produced.

[1]When $E - A$ is odd, $A^2 + E^2$ is also odd.

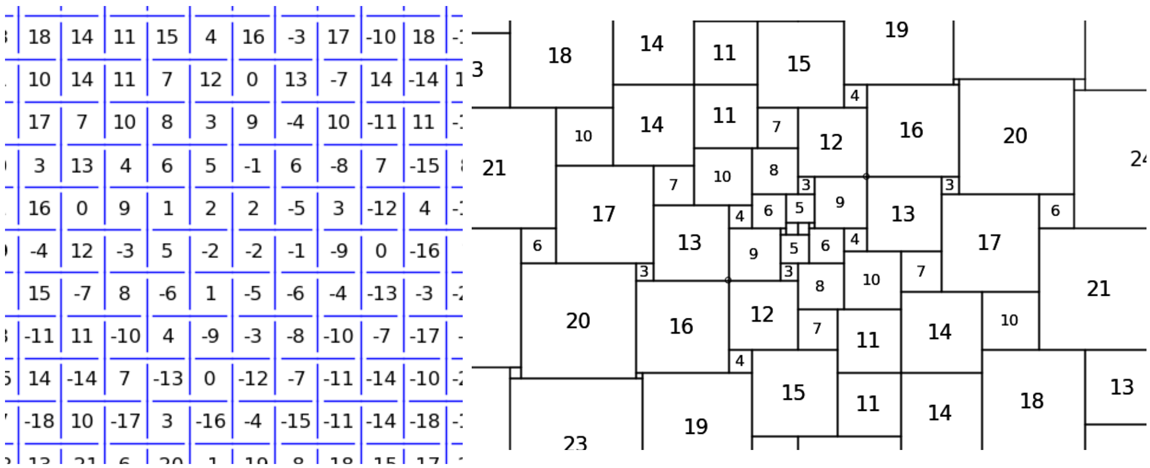


FIGURE 6. A quad-mirror : $\frac{5 \quad -1}{2}\Big|^{6}$.

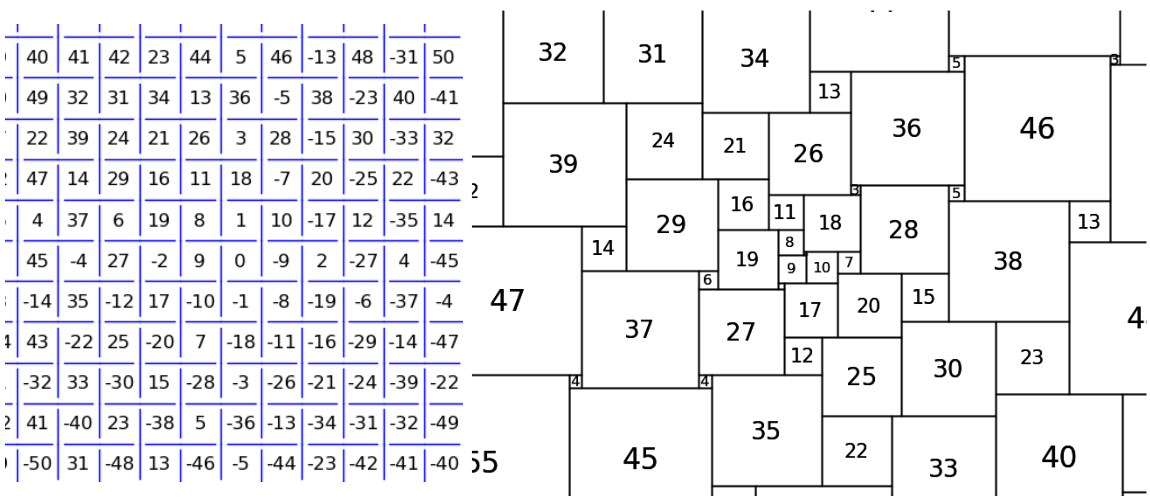


FIGURE 7. A 0-mirror : $\frac{8 \quad 1}{0}\Big|^{10}$

**Conjecture 1.** *If $E-A$ is odd, the grid produced by* $\dfrac{A \quad \frac{E-A+A^2+E^2}{2}}{0}\Bigg|\, E$ *makes a quad-mirror tiling. If $E$ is even, the grid is rotated $90^\circ$ so that $\{A,E\} \to \{E,-A\}$. The quad-mirror is located on the grid at the midpoint of the 0s in the hv-lines* $\dfrac{A \quad \frac{E-A+A^2+E^2}{2}}{0}\Bigg|\, E$ *and* $\dfrac{A \quad \frac{E-A-(A^2+E^2)}{2}}{0}\Bigg|\, E$.

3.6. **Effect of $\dfrac{A^2+E^2}{2}$.** Adding $\pm\dfrac{A^2+E^2}{2}$ to a group, when $A^2+E^2$ is even :

1. 0-mirror grid: the grid is rotated $90°$, and one of the groups is $\times -1$. The tiling is rotated $180°$.

2. quad-mirror grid: move halfway to the $\pm(A^2+E^2)$ hv-line on the same grid.

On quad-mirror grids, E is even. If $\frac{A^2+E^2}{2}$ is an integer, A must also be even. So the grid numbers are either all odd or all even.

3.7. **Semi-antisymmetric flip transformations.**[1]

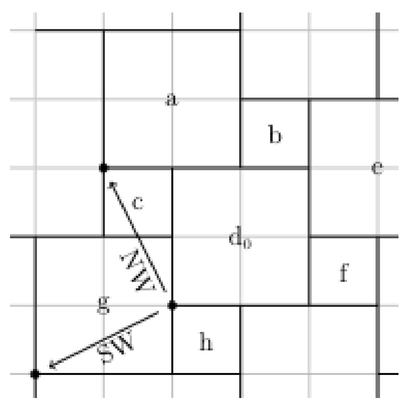
*1. Swap values.* Reverse hv-line values – swap $a$ and $e$, $c$ and $f$ – and multiply one group by $-1$. i.e. $\dfrac{e \quad -b}{-f \quad d}\Big|\begin{matrix} a \\ -c \end{matrix}$ produces the same tiling as $\dfrac{a \quad b}{c \quad d}\Big|\begin{matrix} e \\ f \end{matrix}$, only flipped over.

*2. Swap lines.* Swap all h-lines with v-lines (i.e. every hv-line becomes a vh-line, and vice versa) and multiply one group by $-1$.

## 4. Moving around the tiling

4.1. **Group 1.** I'll measure distances from $d$ in the general hv-line $\dfrac{\overset{+A\nwarrow}{a} \quad \overset{+E\nwarrow\ \nearrow-A}{b}}{c \quad d}\Big|\begin{matrix} \overset{+E\nearrow}{e} \\ f \end{matrix}$ Define the location of each square as its bottom left corner. Starting from square $d_0$, call its bottom left corner $(x_0, y_0)$. (Figure 8.)

Figure 8.




[1] “Semi-antisymmetric” because just one of the groups are multiplied by $-1$.

*1. Moving NW.* In the first hv-line to the NW, $\left[\begin{smallmatrix} +A & +E & +A \\ +E & +A & +E \end{smallmatrix}\right]$, $d_1$ (i.e. $a_0$) is at :

$$x_1 = x_0 - c_0 \qquad\qquad y_1 = y_0 + d_0$$

The second step NW moves to $\left[\begin{smallmatrix} +2A & +2E & +2A \\ +2E & +2A & +2E \end{smallmatrix}\right]$, with $d_2$ at :

$$\begin{aligned} x_2 &= x_1 - c_1 & y_2 &= y_1 + d_1 \\ &= x_0 - c_0 - (c_0 + E) & &= y_0 + d_0 + (d_0 + A) \end{aligned}$$

The third step NW moves to $\left[\begin{smallmatrix} +3A & +3E & +3A \\ +3E & +3A & +3E \end{smallmatrix}\right]$, with $d_3$ at :

$$\begin{aligned} x_3 &= x_2 - c_2 & y_3 &= y_2 + d_2 \\ &= x_0 - c_0 - (c_0 + E) - (c_0 + 2E) & &= y_0 + d_0 + (d_0 + A) + (d_0 + 2A) \end{aligned}$$

After $n$ steps NW, we reach $\dfrac{a_0 + nA \;\; b_0 + nE}{c_0 + nE \;\; d_0 + nA}\Bigg|\begin{matrix} e_0 + nA \\ f_0 + nE \end{matrix}$, with $d_n$ at :

$$\begin{aligned} x_n^{\mathrm{NW}} &= x_0 - c_0 - (c_0 + E) - (c_0 + 2E) & y_n^{\mathrm{NW}} &= y_0 + d_0 + (d_0 + A) + (d_0 + 2A) \\ &\qquad - \cdots - (c_0 + (n-1)E) & &\qquad + \cdots + (d_0 + (n-1)A) \\ &= x_0 - nc_0 - \sum_{k=1}^{n-1} kE & &= y_0 + nd_0 + \sum_{k=1}^{n-1} kA \\ &= x_0 - nc_0 - E\frac{n(n-1)}{2} & &= y_0 + nd_0 + A\frac{n(n-1)}{2} \\ &= x_0 - n\big(c_0 + E\frac{n-1}{2}\big) & &= y_0 + n\big(d_0 + A\frac{n-1}{2}\big) \end{aligned} \tag{1}$$

*2. Moving SW.* The first hv-line to the SW is $\left[\begin{smallmatrix} -E & +A & -E \\ +A & -E & +A \end{smallmatrix}\right]$. $d_1$ is at :

$$\begin{aligned} x_1 &= x_0 - g = x_0 - d_1 & y_1 &= y_0 - h \\ &= x_0 - (d_0 - E) & &= y_0 - (c_0 - E) \end{aligned}$$

since $h$ is $c$ moved SE, i.e. $c - E$ (see Figure 8.)

The second step SW moves to $\left[\begin{smallmatrix} -2E & +2A & -2E \\ +2A & -2E & +2A \end{smallmatrix}\right]$, with $d_2$ at :

$$\begin{aligned} x_2 &= x_1 - d_2 & y_2 &= y_1 - (c_1 - E) \\ &= x_0 - (d_0 - E) - (d_0 - 2E) & &= y_0 - (c_0 - E) - (c_0 + A - E) \end{aligned}$$

The third step SW moves to $\left[\begin{smallmatrix} -3E & +3A & -3E \\ +3A & -3E & +3A \end{smallmatrix}\right]$, with $d_3$ at :

$$\begin{aligned} x_3 &= x_2 - d_3 & y_3 &= y_2 - (c_2 - E) \\ &= x_0 - (d_0 - E) - (d_0 - 2E) & &= y_0 - (c_0 - E) - (c_0 + A - E) \\ &\qquad - (d_0 - 3E) & &\qquad - (c_0 + 2A - E) \end{aligned}$$

So after $n$ steps SW, we reach $\frac{a_0 - nE \;\; b_0 + nA}{c_0 + nA \;\; d_0 - nE} \Big| \frac{e_0 - nE}{f_0 + nA}$, with $d_n$ located at :

$$
\begin{aligned}
x_n^{\mathrm{SW}} &= x_0 - nd_0 + \sum_{k=1}^{n} kE & \qquad y_n^{\mathrm{SW}} &= y_0 - n(c_0 - E) - \sum_{k=1}^{n-1} kA \\
&= x_0 - n\big(d_0 - E\frac{n+1}{2}\big) & &= y_0 - n\big(c_0 - E + A\frac{n-1}{2}\big) \qquad (2)
\end{aligned}
$$

These formulae also work for $n < 0$, i.e. moving NE and SE.

4.2. **Group 2.** Now starting from $c_0$ :

*1. Moving NW.* In the first hv-line to the NW, $\left[\begin{smallmatrix} +A & +E & +A \\ +E & +A & +E \end{smallmatrix}\right]$, $c_1$ is at :

$$
\begin{aligned}
x_1 &= x_0 - c_1 & \qquad y_1 &= y_0 + a_{0\to\mathrm{SW}} \\
&= x_0 - (c_0 + E) & &= y_0 + a_0 - E
\end{aligned}
$$

where $a_{0\to\mathrm{SW}}$ is $a_0$ moved 1 step SW : $a_0 - E$.

In the second hv-line to the NW, $\left[\begin{smallmatrix} +2A & +2E & +2A \\ +2E & +2A & +2E \end{smallmatrix}\right]$, $c_2$ is at :

$$
\begin{aligned}
x_2 &= x_1 - c_2 & \qquad y_2 &= y_1 + a_{1\to\mathrm{SW}} \\
&= x_0 - (c_0 + E) - (c_0 + 2E) & &= y_0 + a_0 - E + a_0 + A - E
\end{aligned}
$$

After $n$ steps NW, $c_n$ is at :

$$
x_{\mathrm{n}} = x_0 - n(c_0 + E\frac{n+1}{2}) \qquad y_{\mathrm{n}} = y_0 + n(a_0 - E + A\frac{n-1}{2})
$$

*2. Moving SW.* In the first hv-line to the SW, $\left[\begin{smallmatrix} -E & +A & -E \\ +A & -E & +A \end{smallmatrix}\right]$, $c_1$ is at :

$$
\begin{aligned}
x_1 &= x_0 - a_1 & \qquad y_1 &= y_0 - c_1 \\
&= x_0 - (a_0 - E) & &= y_0 - (c_0 + A)
\end{aligned}
$$

In the second hv-line to the SW, $\left[\begin{smallmatrix} -2E & +2A & -2E \\ +2A & -2E & +2A \end{smallmatrix}\right]$, $c_2$ is at :

$$
\begin{aligned}
x_2 &= x_1 - a_2 & \qquad y_2 &= y_1 - c_2 \\
&= x_0 - (a_0 - E) - (a_0 - 2E) & &= y_0 - (c_0 + A) - (c_0 + 2A)
\end{aligned}
$$

After $n$ steps SW, $c_n$ is at :

$$
x_{\mathrm{n}} = x_0 - n(a_0 - E\frac{n+1}{2}) \qquad y_{\mathrm{n}} = y_0 - n(c_0 + A\frac{n+1}{2})
$$

4.3. **Moving to any other hv-line.**

*Group 1.* Finding an expression for $d_{new}$, located $m$ steps NW and $n$ steps SW :
1. Moving NW $m$ steps from $d_0$ gets to $\left[\begin{smallmatrix} +mA & +mE & +mA \\ +mE & +mA & +mE \end{smallmatrix}\right]$. From (1), $d_m^{\mathrm{NW}}$ is at :

$$x_m^{\mathrm{NW}} = x_0 - m\big(c_0 + E\frac{m-1}{2}\big) \qquad y_m^{\mathrm{NW}} = y_0 + m\big(d_0 + A\frac{m-1}{2}\big)$$

2. Moving SW $n$ steps from $d_m^{\mathrm{NW}}$ gets to $\dfrac{a_0+mA-nE \;\; b_0+mE+nA}{c_0+mE+nA \;\; d_0+mA-nE}\Bigg|\begin{matrix} e_0+mA-nE \\ f_0+mE+nA \end{matrix}$.
Using (2), with $d_0 \to d_0 + mA$ and $c_0 \to c_0 + mE$, $d_{new}$ is at:

$$\begin{aligned} x &= x_m^{\mathrm{NW}} - n\big(d_0 + mA - E\frac{n+1}{2}\big) \\ &= x_0 - m\big(c_0 + E\frac{m-1}{2}\big) - n\big(d_0 + mA - E\frac{n+1}{2}\big) \end{aligned} \tag{3a}$$

$$\begin{aligned} y &= y_m^{\mathrm{NW}} - n\big(c_0 + mE - E + A\frac{n-1}{2}\big) \\ &= y_0 + m\big(d_0 + A\frac{m-1}{2}\big) - n\big(c_0 + E(m-1) + A\frac{n-1}{2}\big) \end{aligned} \tag{3b}$$

## 4.4. **Location of other squares the same size.**

*Group 1.* Moving NW from $d_0$ $nE$ steps, then SW $nA$ steps, $d_{new}$ is the same size as $d_0$. (Section 2.4).

From (3), with $m \to nE$ and $n \to nA$, $d_{new}$ is at :

$$\begin{aligned} x &= x_0 - nE\big(c_0 + E\frac{nE-1}{2}\big) - nA\big(d_0 + nAE - E\frac{nA+1}{2}\big) \\ &= x_0 - nAd_0 - nE\big(c_0 + E\frac{nE-1}{2} + A\frac{nA-1}{2}\big) \end{aligned} \tag{4a}$$

$$\begin{aligned} y &= y_0 + nE\big(d_0 + A\frac{nE-1}{2}\big) - nA\big(c_0 + E(nE-1) + A\frac{nA-1}{2}\big) \\ &= y_0 + nE\big(d_0 - A\frac{nE+1}{2}\big) - nA\big(c_0 - E + A\frac{nA-1}{2}\big) \end{aligned} \tag{4b}$$

$$\text{hv-line} = \frac{a_0 \quad b_0{+}n(A^2{+}E^2)}{c_0{+}n(A^2{+}E^2) \quad d_0}\Bigg|\begin{matrix} e_0 \\ f_0{+}n(A^2{+}E^2) \end{matrix}$$

## 4.5. **Location of 0-squares in tilings.**

**Conjecture 2.** *All squares of size 0 are collinear in any tiling.*

*I. From 0-sized d to other 0-sized d squares.*
*Case 1.* d=0 and E=0. $\frac{A}{\;\;0}\big|^{0}$ All squares on the central SW-NE diagonal are 0, and $x_n^{\mathrm{SW}} = x_0 - n\big(d_0 - E\frac{n+1}{2}\big)$ collapses into $x_n^{\mathrm{SW}} = x_0$, i.e. they're all on the same vertical line. e.g. $\frac{-1 \;\; 1}{\;\;0}\big|^{0}$ (Figure 10).

*Case 2. d=0, A,E $\neq$ 0.* $\frac{A}{\;\;0}\big|^{E}$

If there's a 0-sized $d$ in a grid, $\frac{A \;\; b}{\;\;0}\big|^{E}$, another one will be found at $\frac{A \;\; b+n(A^2+E^2)}{\;\;0}\big|^{E}$.

When $d_0 = 0$, (4) simplifies to :

$$x = x_0 - nE\big(c_0 + E\frac{nE-1}{2}\big) - nA\big(nAE - E\frac{nA+1}{2}\big)$$

$$= x_0 - nE\big(c_0 + \frac{n(A^2+E^2)-(A+E)}{2}\big) \tag{5a}$$

$$y = y_0 + nE\big(A\frac{nE-1}{2}\big) - nA\big(c_0 + nE^2 - E + A\frac{nA-1}{2}\big)$$

$$= y_0 - nA\big(c_0 + \frac{n(A^2+E^2)-(A+E)}{2}\big) \tag{5b}$$

i.e. the slope $\frac{y-y_0}{x-x_0}$ from any other 0-sized square $d$ to $d_0$ is $A/E$, so they are all on lattice points on the line $y - y_0 = A/E(x - x_0)$ and thus collinear.

## 5. Other tilings

Figure 9: $\frac{1\;2}{1}\big|^{1}$ $(A = E = 0)$ is the tiling with alternating squares of 1 and 2; it doesn't 'bend' around in the way the usual grid does, and tiles the plane once only.

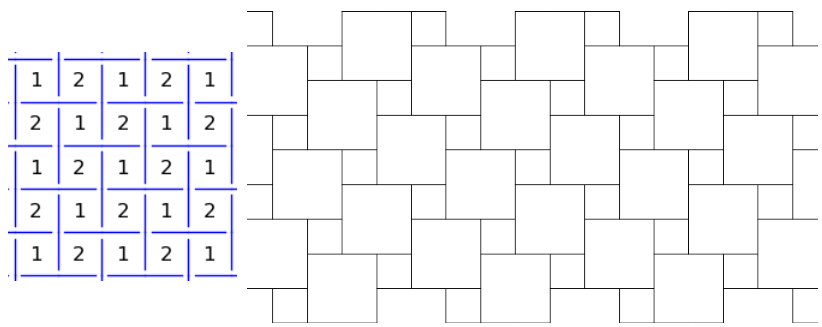



Figure 9. $\frac{1\;2}{1}\big|^{1}$

Figure 10: A quad-mirror hv-line with $E = 0$, $\frac{A\;-A}{0}\big|^{0}$, makes a quad-mirror of zeros, e.g. $\frac{-1\;1}{0}\big|^{0}$.

## 6. Further questions

Q. Prove that no proper tiling contains any Ponting square packing. Is there some other arrangement where a $n \times n$ square in the grid contains $1..n$?

Q. Give the formula for any grid number $(x, y)$, and tiling square size and location, given hv-line data.

Q. There are $d = 0$ squares when, exactly? There are $c = 0$ squares when, exactly? Given that there are $d = 0$ squares, there are $c = 0$ squares when, exactly?

Q. Expanding the grid. Is there some way if putting more numbers between the existing grid cells? e.g. a doubled Ponting packing has $2, 4, 6 \ldots 2n^2$. Is there some way of putting other series in the gaps in between with the odd numbers?

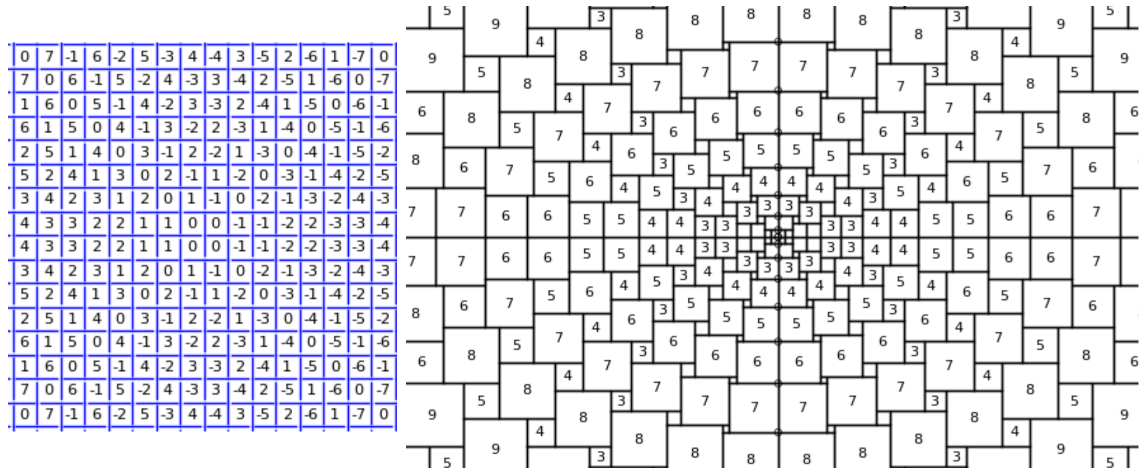



FIGURE 10. $\frac{1\ \ -1}{0}\Big|^{0}$ – a quad-mirror of zeros.

Q. Why does 180° of grid expand to 360° of tiling? Obviously 'double rainbow' looks like $\sqrt{z}$ because both have the angle-halving/doubling thing, rotational squashing. But why does that always happen? They're parabolic. Why? The $x$ and $y$ coordinates always have a non-linear relationship. Why doubling, not tripling? I guess, because of the 2 in $y = x^2$. Can tripling grids be made?

So, draw each tiling with each square covering 1/2 the angle from centre, then 720° will fit in 360°. i.e. draw the square roots of complex coords. That's equivalent to drawing on a double-cartesian plane, with 720° shown in 360. Two of each axis? Then *all* grids will be 'proper'.

Q. Where's the centre, in tilings without a mirror? What's the function returning the coordinates of the centre, given any hv-line of a grid?

Q. What if the edges between the two tiling halves are a straight line? Then they could have different squares fitting together. And wouldnt need a mirror? But I think a straight edge only happens when $A = E$.

Q. Is it sufficient for a proper tiling that the grid has C2 antisymmetry? It seems so. Try to make a proper tiling that has neither 0-mirror nor quad-mirror.

## REFERENCES

[1] Adam Ponting, *Square Packing*, May 2014. http://www.adamponting.com/square-packing/
[2] Adam Ponting, *how to construct double rainbow tesselations*, Dec. 2016. http://www.adamponting.com/construct-rainbow-tesselations/
[3] Ed Pegg Jr, *Ponting Square Packing*, http://demonstrations.wolfram.com/PontingSquarePacking/ Wolfram Demonstrations Project, Published: December 28, 2016

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